ON INFINITE DISCRETE APPROXIMATE SUBGROUPS IN $\mathbb{R}^d$

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Abstract. In this paper we show that any discrete, infinite approximate subgroup $\Lambda \subset \mathbb{R}^d$ is relatively dense around some linear subspace $L \subset \mathbb{R}^d$, i.e., there exists $R > 0$ such that for every ball $B_R(x)$ with center at $x \in L$ we have $\Lambda \cap B_R(x) \neq \emptyset$, and $\Lambda \subset \cup_{x \in L} B_R(x)$. As an application of our main theorem, we provide a complete classification of infinite approximate subgroups in $\mathbb{Z}^d$.

1. Introduction

In this paper we study approximate subgroups. Recall that for a group H[1], a set $\Lambda \subset H$ is called an *approximate subgroup* if there exists a finite set $F \subset H$ such that $\Lambda - \Lambda \subset \Lambda + F$.

Any finite set in a group H is an approximate subgroup. An interesting question of classification of approximate subgroups arises if we control the cardinality of F, while the cardinality of Λ is finite but much larger than of the set of translates F, and in this case we say that Λ has a small doubling. The classification of finite sets having small doubling for the ambient group $H = \mathbb{Z}$ has been obtained by Freiman in his seminal work [2]. These results have been eventually extended to arbitrary abelian groups by Green and Ruzsa [3], and in the case of an arbitrary ambient group by Hrushovski [4], and by Breuillard, Green and Tao [1].

We will investigate here infinite discrete approximate subgroups in $H = \mathbb{R}^d$. Infinite discrete relatively dense approximate subgroups in $\mathbb{R}^d$, *Meyer sets*, have been studied extensively by Meyer [6], Lagarias [5], Moody [7] and many others. It has been proved by Meyer [6] that a discrete relatively dense approximate subgroup $\Lambda \subset \mathbb{R}^d$ is a subset of a model (cut and project) set [7]. Thus, despite a possible aperiodicity of Meyer sets, they all arise from lattices in (possibly) much higher dimensional spaces.

The paper addresses a natural question of what kind of structure has to possess an infinite discrete approximate subgroup Λ in $\mathbb{R}^d$ which is not relatively dense in the whole space. The

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[1]We will be using the additive notation since we are interested in the case where H is commutative.

conclusion that we derive here is that an infinite discrete approximate subgroup in $\mathbb{R}^d$ is almost as rigid as a Meyer set. In particular, we show that any discrete infinite approximate subgroup in $\mathbb{R}^d$ is at the bounded distance from a Meyer set "living" on a subspace of $\mathbb{R}^d$, see Theorem 2.3.

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2. **Main Results**

We will always assume that the underlying group H posses an H-invariant metric d_H, and for any $r > 0$ and $h \in H$ we will denote by $B_r(h) = \{g \in H \mid d_H(g, h) \leq r\}$ the ball of radius r around h. We will call a set $\Lambda \in H$ *relatively dense* if there exists $R > 0$ such that for every $h \in H$ we have $B_R(h) \cap \Lambda \neq \emptyset$. It is easy to see that if an approximate subgroup Λ is discrete then it is also uniformly discrete, i.e., there exists $\delta > 0$ such that for any $h \in H$ the ball of radius δ at h intersects Λ in at most one point.

In this paper we show that discrete approximate subgroups in $\mathbb{R}^d$ are relatively dense around some subspace. Our main result is

Theorem 2.1. Let $\Lambda \subset \mathbb{R}^d$ be an infinite, discrete, approximate subgroup. Then there exists a linear subspace $L \subset \mathbb{R}^d$, and $R > 0$ such that:

- For every $y \in L$ the ball of radius R and center y, i.e., $B_R(y) = \{x \in \mathbb{R}^d \mid \|x-y\| \leq R\}$, intersects non-trivially Λ.
- The R-neighbourhood of L in $\mathbb{R}^d$ contains Λ, i.e.,

$$\Lambda \subset \bigcup_{y \in L} B_R(y).$$

As a corollary, we obtain a complete characterization of infinite approximate subgroups in $\mathbb{Z}^d$.

Theorem 2.2. Let Λ be a subset in $\mathbb{Z}^d$. The set Λ is an infinite approximate subgroup if and only if there exists a linear subspace $L \subset \mathbb{R}^d$ and $R > 0$ such that

- $\Lambda \subset \bigcup_{y\in L} B_R(y)$,

- For every $y \in L$ we have $\Lambda \cap B_R(y) \neq \emptyset$.

As another application of Theorem 2.1, we prove that any discrete approximate subgroup in $\mathbb{R}^d$ is "very close" to be a Meyer set on a subspace of $\mathbb{R}^d$. More precisely, we prove the following result.

Theorem 2.3. Let $\Lambda \subset \mathbb{R}^d$ be an infinite discrete approximate subgroup. Then there exist a subspace $L \subset \mathbb{R}^d$ and $R > 0$ such that:

- The projection Λ_L of Λ on the subspace L is a Meyer set in L, i.e., Λ_L is discrete relatively dense approximate subgroup in L,

- $\Lambda \subset \Lambda_L + B_R(0_{\mathbb{R}^d})$.

3. **Proof of Theorem 2.1**

Let $\Lambda \subset \mathbb{R}^d$ be an approximate subgroup. Then $\Lambda \cup \{0_{\mathbb{R}^d}\}$ is also an approximate subgroup. Therefore without loss of generality we assume that $0_{\mathbb{R}^d} \in \Lambda$. Denote by $K = diam(F)$. The following two important properties will enable us to treat arbitrary approximate subgroup as being almost symmetric:

(**A**) for every $\ell \in \Lambda$ there exists $\ell' \in B_K(-\ell) \cap \Lambda$,
(**B**) for any $\ell_1, \ell_2 \in \Lambda$ there exists $\ell' \in B_{2K}(\ell_1 + \ell_2) \cap \Lambda$.

We will call the property (A) the almost symmetry, and (B) the almost doubling. We start with an easy observation which proves Theorem 2.1 in the case $d = 1$.

Proposition 3.1. Let $\Lambda \subset \mathbb{R}$ be an infinite discrete approximate subgroup. Then Λ is relatively dense.

Proof. Assume that $\Lambda \subset \mathbb{R}$ is an infinite approximate subgroup. Take $\ell \in \Lambda$ with $\ell > 3K$ (which exists by uniform discreteness of Λ). By the almost doubling property there exists $\ell_2 \in \Lambda$ with $\ell_2 \in [2\ell - 2K, 2\ell + 2K] \subset [\ell + K, 2\ell + 2K]$. Similarly, there exists $\ell_3 \in \Lambda \cap B_{2K}(\ell_2 + \ell)$. Therefore, $\ell_3 \in [\ell_2 + K, \ell_2 + \ell + 2K]$. Assume that we already constructed $\ell_1 = \ell, \ell_2, \ldots, \ell_n \in \Lambda$ satisfying that $\ell_m + K \leq \ell_{m+1} \leq \ell_m + \ell + 2K$ for $m = 1, \ldots, n-1$. Then there exists $\ell_{n+1} \in \Lambda \cap [\ell_n + \ell - 2K, \ell_n + \ell + 2K]$. Therefore, we construacted an increasing sequence in $\Lambda \cap \mathbb{R}_+$ with bounded gaps. By almost symmetry property of Λ, we also have in Λ the elements $\{-\ell', -\ell'_2, \ldots, -\ell'_n, \ldots\}$ with $\ell' \in B_K(-\ell)$. This finishes the proof of the Proposition. $\square$

A higher-dimensional case is much more subtle. An important role in the proof of Theorem 2.1 will play the set of asymptotic directions of the points in Λ.

Definition 3.2. Let $\Lambda \subset \mathbb{R}^d$ be a uniformly discrete infinite set. We call

$$D(\Lambda) = \{u \in S^{d-1} \,|\, \text{ there exists } (\ell_n) \in \Lambda \text{ with } \frac{\ell_n}{\|\ell_n\|} \to u \text{ and } \ell_n \to \infty\}$$

the *set of asymptotic directions* of Λ.

It is easy to see that $D(\Lambda)$ is non-empty closed set. It will be very convenient to us to introduce the subspace generated by $D(\Lambda)$. Let $L \subset \mathbb{R}^d$ be the smallest linear subspace with the property that $D(\Lambda) \subset L$. In other words, we have

$$L = Span(D(\Lambda)).$$

The next lemma is an important ingredient in the proof of Theorem 2.1.

Lemma 3.3. Assume that Λ is an infinite discrete approximate subgroup. Let $L = Span(D(\Lambda))$ be a proper subspace in $\mathbb{R}^d$. Then there exists $R > 0$ ($R = 3 \cdot diam(F)$) such that

$$\Lambda \subset \bigcup_{x \in L} B_R(x).$$

Proof. Let $\Lambda \in \mathbb{R}^d$ be an infinite discrete approximate subgroup, i.e., there exists a finite set $F \subset \mathbb{R}^d$ with $\Lambda - \Lambda \subset \Lambda + F$. Denote by $K = diam(F)$. For any $\varepsilon > 0$ and any $u \in S^{d-1}$ we define the cone

$$V_\varepsilon(u) = \{tv \,|\, t > 0, v \in S^{d-1} \text{ with } \langle v, u\rangle \geq 1 - \varepsilon\}.$$

Let us take $R = 3K$. We claim that

$$\Lambda \subset \bigcup_{x \in L} B_R(x).$$

Indeed, if there exists $\ell \in \Lambda$ such that $\ell \notin \bigcup_{x \in L} B_R(x)$, let us define $u = \frac{\ell}{\|\ell\|}$ and $1 - \varepsilon = \frac{\sqrt{\|\ell\|^2 - 5K^2}}{\|\ell\|}$. Then we construct a sequence $\ell_1, \ell_2, \ell_3, \ldots$ in Λ with $\ell_n \to \infty$ and $\ell_n \in V_\varepsilon(u)$. Since, clearly, we have

$$V_\varepsilon(u) \cap L = \{0_{\mathbb{R}^d}\},$$

this will imply the contradiction.

The construction is the same as in the proof of Proposition 3.1. Let us define $\ell_1 = \ell$. We find $\ell_2 \in B_{2K}(\ell_1 + \ell) \cap \Lambda$. The following calculation guarantees that $\ell_2 \in V_\varepsilon(u)$:

$$\left\langle \frac{\ell_2}{\|\ell_2\|}, u \right\rangle \geq \frac{2\|\ell\|}{\sqrt{4\ell^2 + 4K^2}} = \frac{\|\ell\|}{\sqrt{\ell^2 + K^2}} \geq 1 - \varepsilon.$$

Also, it is clear that $\|\ell_2\| \geq \|\ell_1\| + K$. Assume that we constructed a finite sequence $\ell_1, \ell_2, \ldots, \ell_n \in \Lambda$ with $\|\ell_{m+1}\| \geq \|\ell_m\| + K$, $m = 1, \ldots, n-1$, and $\ell_1, \ell_2, \ldots, \ell_n \in V_\varepsilon(u)$. Then there exists $\ell_{n+1} \in B_{2K}(\ell_n + \ell) \cap \Lambda$. Clearly, we have

$$\|\ell_{n+1}\| \geq \|\ell_n\| + K.$$

Finally, for any vector $v \in V_\varepsilon(u)$ we have

$$B_{2K}(v + \ell) \subset V_\varepsilon(u).$$

This will guarantee that $\ell_{n+1} \in V_\varepsilon(u)$. Indeed, if a vector $v \in V_\varepsilon(u)$, then $v + V_\varepsilon(u) \subset V_\varepsilon(u)$, and therefore we have:

$$dist(v + \ell, \partial V_\varepsilon(u)) \geq dist(v + \ell, \partial(v + V_\varepsilon(u)))$$
$$= dist(\ell, \partial(V_\varepsilon(u))) = \|\ell\|(1 - \varepsilon) = \sqrt{\|\ell\|^2 - 5K^2} > 2K.$$

□

Our next step in the proof of Theorem 2.1 is to construct a system of "basis" vectors for Λ. Let $L = Span(D(\Lambda))$, and let R satisfy

$$\Lambda \subset \bigcup_{x \in L} B_R(x).$$

Assume that $dimL = k$, where $1 \leq k \leq d$, and denote by $K = diam(F)$. By uniform discreteness of the approximate subgroup Λ, there exists $\varepsilon > 0$ such that for every $M > 0$ there exist k elements $\ell_1, \ldots, \ell_k \in \Lambda$ satisfying the following properties:

- (ε**-well spreadness**) For all $1 \leq i \leq k$, any $v_i \in B_{2K}(\ell_i)$, and $v_j \in B_{2K}(\varepsilon_j \ell_j), j \neq i, \varepsilon_j \in \{-1, 1\}$, let us denote by γ_i the angle between v_i and the subspace $V^i = Span\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k\}$. Then we require:

$$\varepsilon < \gamma_i < \pi - \varepsilon,$$

- (**no short vectors**) For every $1 \leq i \leq k$ we have

$$\|\ell_i\| \geq M.$$

By almost symmetry of Λ, we can also find the "reflected" vectors $\{\ell'_1, \ldots, \ell'_k\} \subset \Lambda$ which satisfy the property

$$\ell'_i \in B_K(-\ell_i),\ i = 1, \ldots, k.$$

Let us denote by $\mathcal{F} = \{\ell_1, \ldots, \ell_k, \ell'_1, \ldots, \ell'_k\}$. By Lemma 3.1 there exists $R > 0$ such that $\Lambda \subset L_R$, where $L_R = \bigcup_{x \in L} B_R(x)$ the R-thickening of the subspace L. Let us assume that $R \geq K$. Finally, for any choice of $M > 0$, let us call the corresponding system $\mathcal{F}$ as (M, ε, L, R)-system in $\mathbb{R}^d$, and denote by $T(\mathcal{F}) = \max\{\|\ell_i\| \,|\, i = 1, \ldots, k\}$.

Our next claim is the following.

Proposition 3.4. Let $0 < \varepsilon < 1$ and $R, K > 0$. There exists $M > 0$ large enough such that for every (M, ε, L, R)-system $\mathcal{F} = \{\ell_1, \ldots, \ell_k, \ell'_1, \ldots, \ell'_k\}$ in $\mathbb{R}^d$ and for any $x \in \mathbb{R}^d \cap L_R$ with $\|x\|$ large enough there exists $\ell \in \mathcal{F}$ such that for every $v \in B_K(\ell)$ we have

$$\|x - v\| \leq \|x\| - \frac{M\varepsilon}{12}.$$

Proof. By continuity of the distance function, it is enough to prove the conclusion of the proposition in the case where

- $\mathcal{F}$ is symmetric, i.e., if $\ell \in \mathcal{F}$ then $-\ell \in \mathcal{F}$,
- $\mathcal{F} \subset L$,
- $x \in L$,
- $v \in \mathcal{F}$.

Let us call the system $\mathcal{F}$ an (M, ε)-system, since $\mathcal{F}$ is already inside the subspace L. Our next step is to observe that for any vector $x \in S(L) = \{x \in L \,|\, \|x\| = 1\}$ there exists $v \in \mathcal{F}$ such that the angle between x and v, denoted by γ, satisfies:

$$0 \leq \gamma < \pi/2 - \varepsilon/2.$$

Indeed, in the case of $dim(L) = 1$ the statement is obviously true. Assume that we know that the statement is true for the case $dim(L) = k - 1$. Let $dim(L) = k$, and let us define $W = Span\{\ell_1, \ldots, \ell_{k-1}\}$. Let us denote by α_i, $i = 1, \ldots, k-1$, the angles between x and $L_i = \{t\ell_i \,|\, t \in \mathbb{R}\}$. Assume that we have for all $i = 1, \ldots, k-1$:

$$\pi/2 - \varepsilon/2 \leq \alpha_i \leq \pi/2.$$

Then we claim that for every vector $w \in W$, the angle α between x and w satisfies:

$$\pi/2 - \varepsilon/2 \leq \alpha \leq \pi/2.$$

Indeed, denote by $\delta = \cos^{-1}(\pi/2 - \varepsilon/2)$. Then we have for every vector $w = \sum_{i=1}^{k-1} c_i \ell_i \in W$ that

$$\frac{|\langle x, w\rangle|}{\|w\|} = \frac{\left|\sum_{i=1}^{k-1} c_i \langle x, \ell_i\rangle\right|}{\|w\|} \leq \delta \frac{\|w\|_1}{\|w\|_2} \leq \delta.$$

This shows that the angle between x and w satisfies the claim.

Let us denote by $U = Span\{x, \ell_k\}$. If $\dim U = 1$, the claim that $0 \leq \gamma < \pi/2 - \varepsilon/2$ is obvious. So, assume that $\dim U = 2$. Since $W \cap U$ is one-dimensional, there exists $w \in W$ such that $U \cap W = Span\{w\}$. Denote by $L_w = Span\{w\}$. Then we are in the two-dimensional case, i.e., the vectors x, ℓ_k, w lie in the plane $U \cap W$. Let us denite by α the angle between the vector x and L_w. Then α satisfies

$$\pi/2 - \varepsilon/2 \leq \alpha \leq \pi/2.$$

Denote by β the angle between ℓ_k and L_w. Then β satisfies by ε-spreadness of $\mathcal{F}$:

$$\varepsilon < \beta \leq \pi/2.$$

Altogether, this implies that the angle γ between x and the line spanned by ℓ_k satisfies:

$$0 \leq \gamma < \pi/2 - \varepsilon/2.$$

Next, let us consider a triangle with the vertices at the origin, x and at $v \in \mathcal{F}$ with the angle between x and v, denoted by γ, satisfiying:

$$\gamma < \pi/2 - \varepsilon/2.$$

Denote by $D = \|v\|$. We have that $D \leq T(\mathcal{F})$. Also, denote by $\varepsilon' = \cos(\gamma)$. We have that $\varepsilon' \geq \varepsilon/4$. The cosine rule implies

$$\|x - v\|^2 = \|x\|^2 + D^2 - 2xD\cos(\gamma).$$

Notice that if $\|x\| \geq T(\mathcal{F}) \geq \|v\|$ and assume that $\|x\| \geq \frac{4T(\mathcal{F})}{\varepsilon}$ we have:

$$\|x\| - \|x - v\| = \frac{D(2\|x\|\cos(\gamma) - D)}{\|x\| + \|x - v\|} \geq \frac{D(\|x\|\varepsilon/2 - D)}{3\|x\|} \geq \frac{D\|x\|\varepsilon/4}{3\|x\|} = \frac{D\varepsilon}{12} \geq \frac{M\varepsilon}{12}.$$

□

Proof of Theorem 2.1. Assume that $\Lambda \subset \mathbb{R}^d$ is an infinite discrete approximate subgroup satsifying $\Lambda - \Lambda \subset \Lambda + F$ for a finite set F. Denote by $K = diam(F)$ and by $L = Span(D(\Lambda))$. Then by Lemma 3.3 there exists $R > 0$ such that $\Lambda \subset L_R = \bigcup_{x \in L} B_R(x)$. By the discussion above, there exists $\varepsilon > 0$ such that for an arbitrary $M > 0$ there exists (M, ε, L, R)-system $\mathcal{F}$ within Λ. Let us take $M > 0$ so large that the claim of Proposition 3.4 holds true. Let R' be such that for every $x \in L_R$ with $\|x\| \geq R'$ there exists $\ell \in \mathcal{F}$ with the property that for every $v \in B_K(\ell)$ we have:

$$\|x - v\| \leq \|x\| - \frac{M\varepsilon}{12}.$$

We will show that for every $z \in L_R$ we will have $B_{R'}(z) \cap \Lambda \neq \emptyset$. Assume, on the contrary, that there exists $z \in L_R$ such that $B_{R'}(z) \cap \Lambda = \emptyset$ Take minimal $R_2 > R'$ such that $B_{R_2}(z) \cap \Lambda \neq \emptyset$. This means that for every $r < R_2$ we have $B_r(z) \cap \Lambda = \emptyset$, and that there exists $y \in B_{R_2}(z) \cap \Lambda$.

Let us denote by $x = z - y$. Then $\|x\| = R_2$, and therefore there exists $\ell \in \mathcal{F} \subset \Lambda$ such that for every $v \in B_K(\ell)$ we have

$$\|x - v\| \leq \|x\| - \frac{M\varepsilon}{12} < \|x\| = R_2.$$

But, since Λ is an approximate subgroup with $diam(F) = K$, we have that there exists $v \in B_K(\ell)$ such that $y + v \in \Lambda$. This implies:

$$\|z - (y + v)\| < R_2.$$

Therefore, there exists $r < R_2$ such that $B_r(z) \cap \Lambda \neq \emptyset$. So, we get a contradiction. Therefore, indeed, for every $x \in L_R$ we have $B_{R'}(x) \cap \Lambda \neq \emptyset$. This finishes the proof of the theorem.

□

4. Proof of Theorem 2.2

It follows immediately from Theorem 2.1 that if $\Lambda \subset \mathbb{Z}^d$ is an infinite approximate group, then there exists a subspace $L \subset \mathbb{R}^d$ and $R > 0$ such that $\Lambda \subset L + B_R(0_{\mathbb{R}^d})$, and for every $\ell \in L$ we have that $\Lambda \cap B_R(\ell) \neq \emptyset$. Let us call any Λ that satisfies these constraints with respect to a subspace L as being *relatively dense around* L.

On the other hand, assume that $\Lambda \subset \mathbb{Z}^d$ is relativelty dense around a subspace $L \subset \mathbb{R}^d$. We will show that such Λ is necessarily an approximate subgroup.

Indeed, let us first take $R_1 > 0$ with the property[2] that for any point $x \in \mathbb{R}^d$ we have $B_{R_1}(x) \cap \mathbb{Z}^d \neq \emptyset$. Since, for any $\lambda \in \Lambda$ there exists $\ell \in L$ such that $\lambda \in B_R(\ell)$, we have that

[2]We can take any $R_1 > \frac{\sqrt{d}}{2}$.

for any $\lambda_1, \lambda_2 \in \Lambda$ there exist $x_1, x_2 \in \mathbb{Z}^d \cap L + B_{R_1}(0)$ such that

$$\lambda_i \in B_{R+R_1}(x_i), \text{ for } i = 1, 2.$$

Therefore, there exist $f_1, f_2 \in B_{R+R_1}(0) \cap \mathbb{Z}^d$ such that

$$\lambda_i = x_i + f_i, \text{ for } 1 = 1, 2.$$

Also, notice that $x_1 - x_2 \in L + B_{2R}(0)$. Therefore, there exists $\lambda \in \Lambda$ such that $x_1 - x_2 \in B_{3R}(\lambda)$. Thus, there exists $f' \in B_{3R}(0) \cap \mathbb{Z}^d$ such that $x_1 - x_2 = \lambda + f'$. Finally, let us denote by $F = B_{5R+2R_1}(0) \cap \mathbb{Z}^d$ (finite set). Then we have

$$\lambda_1 - \lambda_2 = (x_1 + f_1) - (x_2 + f_2) = (x_1 - x_2) + (f_1 - f_2) = \lambda + (f_1 - f_2 + f') \in \Lambda + F.$$

This finishes the proof of the Theorem.

□

5. Proof of Theorem 2.3

Let Λ be a discrete approximate subgroup in $\mathbb{R}^d$. By Theorem 2.1 we know that there exist a subspace L and $R > 0$ such that Λ is relatively dense around L, i.e., $\Lambda \subset L + B_R(0_{\mathbb{R}^d})$ and for any $x \in L$ we have $B_R(x) \cap \Lambda \neq \emptyset$. Let us denote by π the orthogonal projection from $\mathbb{R}^d$ to L. And let $\Lambda_L = \pi(\Lambda)$.

By linearity of the map π we get that Λ_L is an approximate subgroup. For $\ell_1, \ell_2 \in \Lambda_L$ there exist $\lambda_1, \lambda_2 \in \Lambda$ such that $\ell_i = \pi(\lambda_i), i = 1, 2$. Denote by $L^\perp$ the orthgonal complement to L, i.e., we have $\mathbb{R}^d = L \oplus L^\perp$. Then there exist $\mu_1, \mu_2 \in L^\perp$ such that

$$\lambda_i = \ell_i + \mu_i, \text{ for } i = 1, 2.$$

But Λ is an aproximate subgroup. Therefore, there exists a finite set $F \subset \mathbb{R}^d$ such that $\Lambda - \Lambda \subset \Lambda + F$. This implies that there exist $\lambda \in \Lambda$, and $f \in F$ such that

$$\lambda_1 - \lambda_2 = \lambda + f.$$

By projecting both sides on L we obtain:

$$\ell_1 - \ell_2 = \pi(\lambda) + \pi(f).$$

Let us denote $F' = \pi(F)$ (a finite set). Then we have

$$\Lambda_L - \Lambda_L \subset \Lambda_L + F'.$$

We also have that Λ_L is relatively dense in L since $\Lambda \subset L + B_R(0_{\mathbb{R}^d})$.

The set Λ_L is discrete. Indeed, assume that it is not discrete. Then there exists $(\ell_n) \subset \Lambda_L$ with $\ell_n \to x \in L$ and $\ell_n \neq x$ for every n. Let $(\mu_n) \subset L^\perp$ such that $\lambda_n = \ell_n + \mu_n \in \Lambda$.

Since all μ_n are bounded, then there is a convergent subsequence (μ_{n_k}). Denote its limit by $\mu \in L^{\perp}$. Then we have

$$\lambda_{n_k} = \ell_{n_k} + \mu_{n_k} \to x + \mu.$$

Since Λ is discrete, this implies that the sequence λ_{n_k} is fixed for k large enough. This implies that the subsequence ℓ_{n_k} is fixed for k large enough and we get a contradiction.

All this together, shows that the set $\Lambda_L \subset L$ is a Meyer set.

Finally, by the construction we have $\Lambda \subset \Lambda_L + B_R(0_{\mathbb{R}^d})$.

□

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